Lecture 9. **Stability of Dynamic Systems across Lyapunov. Lyapunov's theorems on first approximation (the first method of Lyapunov)**

9.1 Stability of Dynamic Systems across Lyapunov

Now, let a dynamic system state be defined as a set of independent coordinates $x_1(t),\ x_2(t),\ \dots,\ x_n(t)$. The system movement is described as a set of rules $x_{10}(t),\ x_{20}(t),\ \dots,\ x_{n0}(t)$. This predefined movement is called *Undisturbed Movement.* Application of external disturbances will cause deflection of real behavior from the predefined one:

$$x_1(t) \neq x_{10}(t),\ x_2(t) \neq x_{20}(t),\ \dots,\ x_n(t) \neq x_{n0}(t).$$

The real behavior is called *Disturbed Movement.*

Definition: predefined undisturbed motion of a system is called *Stable Motion* if as a result of application of external forces (which are eliminated after that) the system disturbed motion after some period of time move to a region

$$\|x_i(t) - x_{i0}(t)\| < \xi_i \quad \forall \xi_i > 0,\ \xi_i = const;\ \left(i = \overline{1,n}\right).$$

Let us now consider stability even more deeply. For the first time strict definition of stability was given by Russian scientist A. M. Lyapunov in 1892.

A little bit mathematics: let a dynamic system be described by a set of nonlinear differential equations in Cauchy form:

$$\frac{dx_i}{dt} = F_i(x_1,\ x_2,\ \dots,\ x_n);\ \forall i == 1,\dots,n;\quad x = \begin{vmatrix} x_1 \\ x_2 \\ \dots \\ x_n \end{vmatrix}; \tag{3.1}$$

$$\dot{x} = F(x).$$

Here $\dot{x} = \begin{vmatrix} \dot{x}_1 \\ \dot{x}_2 \\ \dots \\ \dot{x}_n \end{vmatrix};\quad F(x) = \begin{vmatrix} F_1 \\ F_n \end{vmatrix}$ are nonlinear vector functions of vector argument.

If initial values x_{i0} are given at $t = t_0$ the solution can be written as

$$x_i = x_i(x_{10},\ x_{20},\ \dots,\ x_{n0})^T,\ i = \overline{1,n}.$$

Let the system steady state be described by coordinates $x_i^0 = \left(x_1^0,\ ...,\ x_n^0\right)^T$, $\left(i = \overline{1,n}\right)$. Define coordinates deflection $\Delta x_i = x_i - x_i^0$ $\left(i = \overline{1,n}\right)$ characterizing the deflection of real behavior from undisturbed steady state behavior. Then we can rewrite equations (3.1) in terms of these deflections:

$$\frac{d\Delta x_i}{dt} = f_i\left(\Delta x_1,\ \Delta x_2,\ ...,\ \Delta x_n\right),\ \left(i = \overline{1,n}\right) \qquad (3.2)$$

where f_i are some nonlinear functions.

Equations (3.2) are called *disturbed motion equations***.** Initial values of deflections Δx_{i_0}, $\left(i = \overline{1,n}\right)$ are called *disturbances*. The solution of (3.2) at particular $\Delta x_i = \Delta x_i\left(\Delta x_{10},\ \Delta x_{20},...,\ \Delta x_{n0},t\right)$ is a *disturbed motion***.**

A.M. Lyapunov in his works gave the following definitions of stability.

The first definition: Undisturbed motion $\Delta x_i = 0$ is called *Stable according to Lyapunov* with respect to variables x_i if at any given positive infinitesimal $\xi > 0$ there exist positive $\gamma > 0$ such that for all Δx_{i0} if

$$\left\|\Delta x_{i0}\right\| < \gamma_i \qquad (3.3)$$

then disturbed motion (3.2) for $t > t_0$ satisfies

$$\left\|\Delta x_i\right\| < \xi_i,\ \ i = \overline{1,n} \qquad (3.4)$$

Here norm is: $\left\|\Delta x_i\right\| = \sqrt{\sum_{i=1}^{n} \Delta x_i^2}$.

The second definition: Undisturbed motion is called *Asymptotically Stable according to Lyapunov* if additional condition

$$\lim_{t \to \infty} \Delta x_i(t) = 0 \quad \left(i = \overline{1,n}\right) \qquad (3.5)$$

is satisfied.

The third definition: Undisturbed motion is called *Unstable according to Lyapunov* if there exists moment of time $t = t_1 > t_0$ at which condition (3.4) is not satisfied, i.e.

$$\left\|\Delta x_i\right\| \geq \xi_i \quad \left(i = \overline{1,n}\right).$$

Geometrical interpretation

Figures 3.4 – 3.7 give graphical presentation of notion mentioned above.

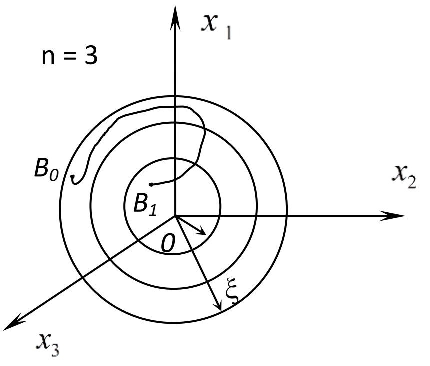

Fig. 3.2a. Stable motion

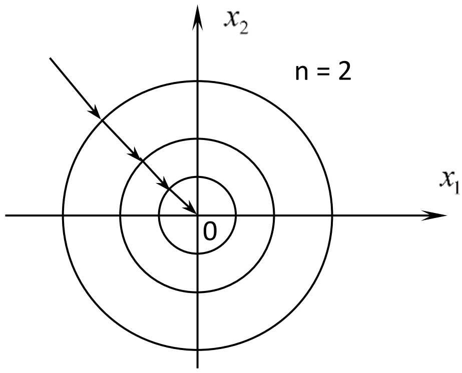

Fig. 3.2b. Asymptotically Stable motion

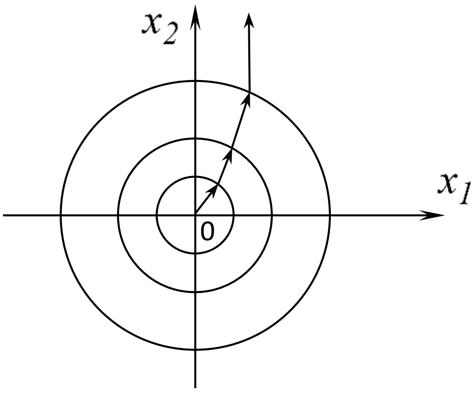

Fig. 3.2c. Unstable motion

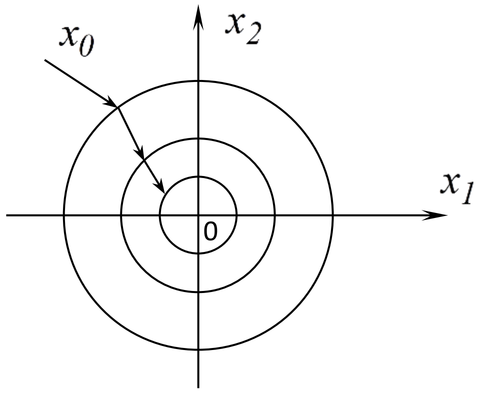

Fig. 3.2d. Stable motion

Physical meaning

Simply stated, linear system is stable if its response to any finite action is also finite, and is unstable if the response is infinite. Stability is essential for normal operation of dynamic control systems.

9.2. Stability to first approximation: A.M. Lyapunov theorems

Let ACS be described in state space by differential equations of the form

$$\frac{dx_i}{dt} = F_i\left(x_1,\ x_2,\ \ldots,\ x_n\right). \qquad (3.6)$$

Here $\left(x_1,\ x_2,\ \ldots,\ x_n\right)$ is a deflection vector.
Motion of the system is (all according to A.M. Lyapunov):

- *Stable* if $\left\|x_{i0}\right\| < \gamma_i$; $\left\|x_i\right\| < \xi_i$;
- *Asymptotically Stable* if $\lim\limits_{t\to\infty} x_i(t) = 0$;
- *Unstable* if $\|\ x_i\| \ \geq \xi_i \quad \forall i = \overline{1,n}$.

Geometrical interpretation in 3-dimensional coordinate space: motion is stable if at disturbances that have not put point $B_0(x_{10}, x_{20}, x_{30})$ out of sphere of radius γ the disturbed motion does not move out of sphere of radius ξ.

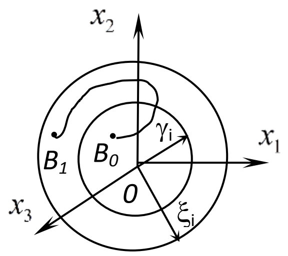

Fig. 3.3. Stable motion

Now, let a dynamic system be presented as a set of equations (3.1). If nonlinear functions $F_i(x_1, x_2, ..., x_n)$ in can be rewritten as convergent power series (Taylor series) then equations of the form (3.6) after such transformation become

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n + R_i(x_1, ..., x_n),$$

where $R_i(x_1, ..., x_n)$ are functions not containing terms of order lower than 2. If deflections are small enough we can neglect R_i thus obtaining linearized equations, which are called first approximation equations:

$$\frac{dx_i}{dt} = a_{i1}x_1 + a_{i2}x_2 + ... + a_{in}x_n, \ i = \overline{1,n} \qquad (3.7)$$

or equivalently in matrix form $\dot{X} = AX$. Why not $\dot{X} = AX + BU$? It is obvious: since we are dealing with stability $U(t) \equiv 0$.

Characteristic equation in this case is

$$det(A - \lambda I) = 0, \qquad (3.8)$$

where λ are proper numbers of matrix A (roots of characteristic equation):

$$det\begin{vmatrix} (a_{11}-\lambda) & a_{12} & ... & a_{2n} \\ a_{21} & (a_{22}-\lambda) & ... & a_{2n} \\ ... & ... & ... & ... \\ a_{n1} & a_{n2} & ... & (a_{nn}-\lambda) \end{vmatrix} = 0; \ \lambda_i = \alpha_i \pm j\beta_i.$$

For stability analysis two theorems have fundamental importance.

THE THEOREM 1. If real parts of all roots λ_i $\left(i=\overline{1,n}\right)$ of characteristic equation (3.8) are negative then undisturbed motion $\Delta x_i = 0$ is *Asymptotically Stable*, i.e. if $\operatorname{Re}\lambda_i(A) < 0 \quad \forall i = \overline{1,n}$ then $\lim\limits_{t\to\infty} \Delta x_i(t) = 0$.

THE THEOREM 2. If among all roots λ_i of characteristic equation (3.8) there is at least one root with positive real part then undisturbed motion is *Unstable*, i.e. if $\operatorname{Re}\lambda_k(A) > 0$ then $\operatorname{Re}\lambda_i(A) < 0$ *(k ≠ i)* despite the fact that $\left\|\Delta x_{i0}\right\| < \varepsilon_i \quad \forall i = \overline{1,n}$.

Important note: if among all roots there are several (possibly one) zeroes, and all other roots have negative real parts then such case is called critical; the system is called neutrally stable. In critical case stability of undisturbed motion cannot be evaluated by first approximation (3.7).

So, according to Lyapunov theorems we can state the following:

1) A linear system movement is *stable* and this *stability is asymptotical* if all the roots of characteristic equation are negative, i.e. $\operatorname{Re}\lambda_i(A) < 0 \quad \forall i = \overline{1,n}$.

2) A linear system movement is *unstable* if among all roots in its characteristic equation there is at least one root with positive real part, i.e. $\operatorname{Re}\lambda_i(A) < 0 \quad \forall i = \overline{1,n-1};\ \operatorname{Re}\lambda_k(A) > 0.$

3) A linear system movement is *called neutrally stable* if among the roots of its characteristic equation there is one zero root and the rest of roots have negative real parts: $\operatorname{Re}\lambda_i(A) < 0 \quad \forall i = \overline{1,n-1};\ \operatorname{Re}\lambda_k(A) = 0$, $i \neq k$.

These theorems are very important since they allow to judge about nonlinear system stability analyzing fairly simple linearized equations and not to mess with complicated nonlinear equations. These theorems form the first stability analysis method of Lyapunov.

9.3 Finding proper numbers and eigenvectors of a matrix

Let us consider a given matrix *A* and some vector V that satisfies an equation:

$$AV = \lambda V .$$

Here scalar variable λ is called *proper number* of *A*, and V is called *eigenvector* (or, equally, *proper vector*) of matrix *A.*

Let us solve the task of finding a proper number and eigenvector. After transposing all terms to the left-hand side and factoring out V we will obtain:

$$(A - \lambda I)V = 0. \tag{3.9}$$

Here *0* denotes a zero vector, i.e. $0 = \begin{vmatrix} 0_1 \\ 0_2 \\ \dots \\ 0_n \end{vmatrix}$.

For equation (3.9) to have solution it is necessary that the determinant of *(A – λI)* to be equal to zero:

$$det\,(A - \lambda I) = 0 \;. \qquad (3.10)$$

Solving (3.10) for λ we obtain polynomial equation of the form:

$$\lambda^n + a_n \lambda^{n-1} + a_{n-1}\lambda^{n-2} + \ldots + a_2\lambda + a_1 = 0.$$

This equation is called *characteristic equation* of the matrix *A*; a_i are characteristic polynomial coefficients. Roots λ_1, λ_2, ..., λ_n of equation (3.10) are proper numbers of the matrix *A*.

After all this theory the time for a good example has come.

An example 1. Let mathematical description of a dynamic system to be given as $\dot{X} = AX + BU$, where $A = \begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix}$; *U(t)=0.*

The task is to find proper numbers of matrix *A*.

Algorithm and Solution

1) Write a characteristic equation of the system

$$det\,(A - \lambda I) = 0\,.$$

2) We will define own numbers of the given matrix *A*:

$$det\left(\begin{vmatrix} 1 & 3 \\ 7 & 5 \end{vmatrix} - \lambda \begin{vmatrix} 1 & 0 \\ 0 & 1 \end{vmatrix}\right) = 0$$

$$det\left(\begin{vmatrix} (1-\lambda) & 3 \\ 7 & (5-\lambda) \end{vmatrix}\right) = 0$$

$$\lambda^2 - 6\lambda - 16 = 0$$

$$\lambda_1 = -2,\; \lambda_2 = 8.$$

Conclusion: according to the theorem the 2nd movement of the researched system is unstable as $\lambda_2 > 0$.

Geometrical interpretation (fig. 3.4):

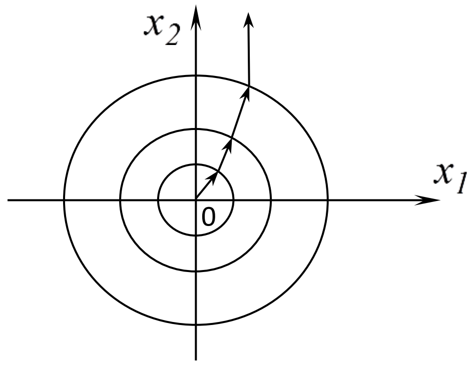

Fig. 3.4. Unstable motion

Vectors V_i corresponding to λ_i are eigenvectors of A, they are determined by the equation

$$(A - \lambda_i I)V_i = 0$$

or equally $$\lambda_i V_i = A V_i \ \forall i = \overline{1,n}. \qquad (3.11)$$

If a particular matrix A has *"n"* different proper numbers, then its corresponding *"n"* proper vectors are *linearly independent.* Since the matrix $(A - \lambda_i I)$ in equation (3.11) has rank not more than *(n-1)* each of V_i is determined accurate to arbitrary multiplier $\forall i = \overline{1,n}$.